ECON 7010 - Macroeconomics I

Fall 2015

Notes for Lecture #6

Today:

- HH Problem
  - Infinite horizon, stochastic income savings problem
  - Endogenous labor supply
  - Portfolio choice

Infinite Horizon Household Problem:

- Sequence Problem (not stochastic)
  - $\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$
  - s.t. $\sum_{t=0}^{\infty} \frac{c_t}{R_t} \leq \underbrace{\sum_{t=0}^{\infty} \frac{y_t}{R_t} + A_0}_{\text{present discounted value of lifetime income}}$
  - $R \equiv$ fixed growth rate (real interest rate)
  - $\Rightarrow$ FOC: $u'(c_t) = \beta R u'(c_{t+1})$
- Dynamic Programming Problem (DPP) Version
  - $(y_t = y \; \forall t) \rightarrow$ making y_t constant
  - $V(A) = \max_{A'} u\left(A + y - \frac{A'}{R}\right) + \beta V(A') \; \forall A$
  - transition equation: $A' = R(A + y - c)$
    - What you have next period is a result of what you come into this period with, what you earn, and what you consume
  - $\Rightarrow A' = \phi(A)$ is the policy function
  - we need a bound: $\underline{A} \leq A' \leq \bar{A}$ (i.e., a borrowing restriction that says can't accumulate debt forever)

Stochastic Income

- y is random - it follows a first-order Markov process (w/ transition matrix Π)
- stationarity (probability from going from one state to another always the same)
- $V(A, y) = \max_{A'} u\left(A + y + \frac{A'}{R}\right) + \beta E_{y'|y} V(A', y'), \forall (A, y)$
- Policy Function: $A' = \Gamma(A, y) \Rightarrow c = \phi(A, y)$
  - This is the modern theory of consumption
  - consumptions depends upon the state - in the stochastic case, future consumption is a random variable
- FOC:

- $u'(c) = \beta R E_{y'|y} V_1(A', y')$
- $V_1(A, y) = u'(c)$, by applying the envelope condition
- $\implies V_1(A', y') = u'(c')$
- $\Rightarrow u'(c) = \beta R E_{y'|y} u'(c')$
- or $u'(\phi(A, y)) = \beta R E_{y'|y} u'(\phi(A', y'))$

- An example: Hall (*JPE*, 1978) (also, Hansen and Singleton (*Econometrica*, 1982))
  - $u'(c_t) = \beta R E_t u'(c_{t+1})$ (expectation notes that using period t information)
  - Hall assumes:
    1. $\beta R = 1 \Rightarrow u'(c_t) = E_t u'(c_{t+1})$
       * $\Rightarrow$ marginal utility follows a random walk
       * Random walk: $x_{t+1} = x_t + \tilde{\varepsilon}_{t+1}$, where $\tilde{\varepsilon}_{t+1} \sim$ iid and $E_t \tilde{\varepsilon}_{t+1} = 0$
       * $\Rightarrow E_t x_{t+1} = x_t$
       * i.e., continue on path with randomizations off of it - like a drunk walking (any deviation has a permanent effect)
    2. $u(c)$ quadratic (e.g., $u(c) = \frac{-1}{2}(\bar{c} - c)^2$) $\rightarrow$ consumption follows a random walk since $u(\cdot)$ follows a random walk, so does c
       * $u(c) = -\frac{1}{2}(\bar{c} - c)^2 \implies u'(c) = (\bar{c} - c) \implies u'(c) = E_t u'(c') = E_t(\bar{c} - c_{t+1}) \implies \bar{c} - c_t = \bar{c} - E_t c_{t+1} \implies c_t = E_t c_{t+1}$
       * $E_t c_{t+1} = c_t$
       * $\Rightarrow c_{t+1} = c_t + \tilde{\varepsilon}_{t+1}$ where the random term comes in because, for example, you can't forecast income 100% and where $E_t \tilde{\varepsilon}_{t+1} = 0$
  - What Hall does is runs the regression: $c_{t+1} = \alpha_0 + \alpha_1 c_t + \alpha_2 z_t + \tilde{\varepsilon}_t$
    * $\tilde{\varepsilon}_t$ is an error term, representing the shocks to income in the theory model
    * z_t is something (anything) known in period t
  - Theory says $\alpha_2 = 0$
    * You shouldn't be able to have anything else (that you know) predict consumption in the future after controlling for present consumption
    * If $\alpha_2 \neq 0$, then the FOC's would not be satisfied and you could rearrange consumption to make yourself better off.
    * We have fixed c_t thinking we know the future $\Rightarrow c_t$ already has all currently available info in it that we base future consumption on.
    * This is because: $E_t c_{t+1} = c_t \Rightarrow c_{t+1} = c_t + \tilde{\varepsilon}_{t+1}$ and $\tilde{\varepsilon}_{t+1}$ is mean 0
  - Hall finds that $\alpha_1 \approx 1$, as theory expects (one for one movement in consumption today and tomorrow)

<u>Adding Endogenous Labor Supply</u>

- w is the stochastic real wage
- period utility $u(c, n)$, where $c =$ consumption, $n =$ labor supply ($n = 1 -$ leisure - where normalization is that time $=1$)
- Static problem:
  - $\max_{c,n} u(c, n)$, s.t. $c = wn + A$
  - or $\max_n u(wn + A, n)$
  - FOC implies: $wu_1 + u_2 = 0$, where $u_1 = u_c(c, n)$ and $u_2 = u_n(c, n)$

- The FOC says that is the marginal disutility of working will exactly offset the marginal utility of consumption times the wage rate (which, since the price of consumption is normalized to \$1, is the marginal utility of consumption per hour worked).

- Dynamic Problem:
  - $V(A,w) = \max_{A,n} u(\underbrace{A + wn - \frac{A'}{R}}_{=c}, n) + \beta E_{w'|w} V(A', w')$
  - Transition equation: $c = A + wn - \frac{A'}{R}$
  - FOCs (2 because 2 control variables):
    1. $\frac{\partial V}{\partial n} = wu_1(c,n) + u_2(c,n) = 0$ (intratemporal condition)
    2. $\frac{\partial V}{\partial A'} = u_1(c,n) - \beta R E_{w'|w} u_1(c', n') = 0 \Rightarrow u_1(c,n) = \beta R E_{w'|w} u_1(c', n')$ (intertemporal)
  - Labor supply and savings decisions <u>are not</u> independent.
    - Can't decide how much to consume and how much to work separately
    - Both FOCs used to derive both policy functions
  - Policy Functions:
    - $n = z(A,w)$
    - $A' = \Gamma(A,w) \Rightarrow c = \phi(A,w)$
    - "calculate" $\frac{\partial n}{\partial w} \rightarrow$ can't take single equation to calculate partial derivative - need to see how change in w impacts n directly then how this affects c and then indirectly affects n (income vs. subs effects)
  - Alternative specification:
    - We can use the fact that to labor decision is not dynamic, although it does depend upon the choice of c/A'.
    - We can thus rewrite the problem as a choice of just one control variable, A'.
    - To see this, start with the FOC for the choice of labor: $wu_1(c,n) = -u_2(c,n)$
    - Plug in the policy function for n, $n = z(A,w)$ and the budget constraint to find: $wu_1(A + wz(A,w) - \frac{A'}{R}, z(A,w)) = -u_2(A + wz(A,w) - \frac{A'}{R}, z(A,w))$
    - This is an equation in terms of the state variables A, A', w. We can thus write the choice of n that satisfies this equation for a given A' as: $n = \psi(A, w, A')$
    - Using this policy function for labor supply, we can rewrite the Bellman equation as:
    - $V(A,w) = \max_{A'} u(A + w\psi(A,w,A') - \frac{A'}{R}, \psi(A,w,A')) + \beta E_{w'|w} V(A', w')$
    - Now we've reformulated the problem so that there is just one choice variable, A'
    - The FOC will yield: $\frac{-1}{R} u_1(A + w\psi(A,w,A') - \frac{A'}{R}, \psi(A,w,A')) + u_1(A + w\psi(A,w,A') - \frac{A'}{R}, \psi(A,w,A')) \frac{\partial \psi(A,w,A')}{\partial A'} + u_2(A + w\psi(A,w,A') - \frac{A'}{R}, \psi(A,w,A')) \frac{\partial \psi(A,w,A')}{\partial A'} + \beta u_1(A' + w\psi(A',w',A'') - \frac{A''}{R'}, \psi(A',w',A'')) = 0$
    - One can do comparative statics with this equation in the usual fashion.
    - This equation gives the defines the policy function for savings, $A' = \Gamma(A,w)$. Once one solves for this, one can use $n = \psi(A,w,A')$ to solve for labor supply

<u>Household Portfolio Problem</u>

- Portfolio problem (i.e., What is the optimal portfolio of assets to hold?)
  - N-assets, random returns
  - use past returns to predict future returns
  - We'll focus on this problem

- This problem forms the basis for asset pricing theory (see, Sargent and Lj Chap 3, Lucas (Econometrica 1978))
- Problem:
  - state: (A, y, R_{-1}), where R_{-1} is a vector of length N with past returns by asset
  - control: asset purchases: $\{s_i\}_{i=1}^N \rightarrow$ amount invest in each asset, can't control future asset amounts because returns are random
  - constraint: $c + \sum_{i=1}^N s_i = A + y$ (per period budget constraint)
  - transition equation: $A' = \sum_{i=1}^N R_i s_i$
  - Functional equation: $V(A, y, R_{-1}) = \max_{\{s_i\}_{i=1}^N} u(A+y-\sum_{i=1}^N s_i) + \beta E_{y',R|y,R_{-1}} V(\sum_{i=1}^N R_i s_i - y', R)$
  - Need to be specific about when you know things. In the above model, the agent knows current income, but not the current return when making the consumption decision.
  - Draw a timeline with hold events unfold: 1) know (A, y) 2) decide on s_i 3) Returns realized 4) start new period (returns imply A')
  - There are N FOCs:
    * $u'(c) = \beta E_{(\cdot)} V_1(\cdot) R_i, i = 1, ..., N$ (these are from $\frac{\partial V}{\partial s_i}$)
    * $\Rightarrow u'(c) = \beta E_{(\cdot)}\{u'(c')R_i\}, i = 1, ..., N \rightarrow$ Euler equations
    * This means: $E_{(\cdot)} u'(c')R_i = z, \forall i$, where z is some constant (this means $z = u'(c)$ not change with i)
    * Using the rule that $E(XY) = E(X)E(Y) + cov(X, Y)$ we can write:
    * $\Rightarrow E_{y',R_i|y,R_{i,-1}} u'(c') E_{R_i|R_{i,-1}} R_i + \underbrace{cov(u'(c'), R_i)}_{\text{related to Capital Asset Pricing Model (CAPM)}} = z, \forall i$
    * Since z is the same for all i, it must be the case that those assets with a higher expected returns (ER_i) must have a lower covariance between their return and marginal utility.
    * Since marginal utility and the level of consumption are inversely related, this is the same as saying that assets with a higher return much have a higher covariance with consumption.
    * The relation to the CAPM comes from this fact: CAPM says that only assets with undiversifiable risk get excess returns (i.e., you get excess returns only on stocks that are highly correlated with aggregate income $\rightarrow \beta$). That's what these equation describing the optimal choice of asset holdings are saying .
    * Empirical asset pricing tests for efficient markets by estimating an equation based on these necessary conditions. The standard CAPM regression is: $R_{i,t} - R_{f,t} = \alpha_i + \beta_i(R_{m,t} - R_{f,t}) + \varepsilon_{i,t}$
    * Where β_i is the a function of the covariance between the assets return and the market return (i.e., a measure of the assets systemic risk).
    * Recall that $\beta_i = \frac{cov((R_{m,t}-R_{f,t}),(R_{i,t}-R_{f,t}))}{var(R_{m,t}-R_{f,t})}$
    * The intercept term, α_i represents the excess return to asset i. If markets are efficient, only the β_i should be significantly different from zero.

Example: Hansen-Singleton (Econometrica 1982)

- Hall assumed $\beta R = 1$, quadratic utility and tested the random walk theory of consumption.
- Hansen and Singleton assume $u(c) = \frac{c^{1-\gamma}}{\gamma}$ and stochastic returns
  - They estimate (β, γ) from the Euler equations using current information (they do this for all i Euler equations)
    * they have $i = 2$ (a risk free and a overall stock market return)

        - $\Rightarrow$ 2 equations, 2 unknowns
        - They find that $\beta \approx 0.99$, $\gamma \approx 0.67$ to 0.95
    - H&S pick a family of utility functions and assume $E_t\varepsilon_{t+1} = 0$ and find β and γ that makes this happen.

- Hall did:
    - $u'(c) = Eu'(c')$
    - $u'(c_t) = E_t u'(c_{t+1})$
    - $u'(c_t) = u'(c_{t+1}) + \varepsilon_{t+1}$
    - and has $E_t\varepsilon_{t+1} = 0$ as a restriction

- H-S do:
    - $u'(c) = \beta Eu'(c')R_1$
    - $u'(c) = \beta Eu'(c')R_2$
    - $\Rightarrow u'(c_t) = \beta Eu'(c_{t+1})R_1 + \varepsilon_{1,t+1}$
    - $\Rightarrow u'(c_t) = \beta Eu'(c_{t+1})R_2 + \varepsilon_{2,t+1}$
    - $\Rightarrow \beta\left(\left[\frac{c_{t+1}}{c_t}\right]^{-\gamma} R_{i,t+1}\right) - 1 = \varepsilon_{i,t+1}, \forall\ i$
    - and have $E_t\varepsilon_{i,t+1} = 0$ as a restriction

- $N = 2$ case:
    - $R_1 = \bar{R}$ wp 1
    - $R_2 =$ random variable
    - $\Rightarrow \bar{R}Eu'(c') = ER_2Eu'(c') + \underbrace{cov(R_2, u'(c'))}_{\text{wouldn't expect this to go to zero(stocks and cons move together)}}$
    - When is $s_1 > 0$ and $s_2 > 0$? (i.e., the agent holds some of each asset)
    - For this to be the case, the FOC will hold with equality. What does this imply about returns?
    - Is $\bar{R} < ER_2$? or $\bar{R} > ER_2$? (i.e., is the risk free return greater than or equal to the return on the risk free asset?)
    - Suppose, $\bar{R} > ER_2$? Then need $cov(R_2, u'(c')) > 0$ (b/c $\bar{R}Eu'(c') = ER_2Eu'(c') + cov(R_2, u'(c'))$), which is not the case (else it'd be insurance not equity invest) (notes $cov(R_2, u'(c')) > 0$ means you get a higher return when the MUC is higher)
    - Thus, if $cov(R_2, u'(c')) < 0$, we know that $\bar{R} < ER_2$ - this says that you need a higher return to hold the risky asset than to hold the risk free asset
    - Again, this is the basis the CAPM model - and asset pricing models generally - return on risky assets is some premium over the risk free asset that varies depending how correlated the asset is with systematic risk (also called market risk).